The Method of Ascent and $\cos\left(\sqrt{A^2+B^2}\,\right)^*$

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1 Introduction

The well-known Hadamard method of descent [2, 3] consists of deriving solutions of wave equations in n variables from (known) solutions in more than n variables. To quote Hadamard, (the method) "consists in noticing that he who can do more can do less". It is the purpose of this paper to exhibit cases in which he who can do less can do more — that sometimes one can build the propagator $\cos\left(\sqrt{A^2+B^2}\,t\right)$ out of $\cos(At)$ and $\cos(Bt)$.

In Section 2 we consider the commutative case and represent $\cos\left(\sqrt{A_1^2+\cdots+A_n^2}\,t\right)$ in terms of an integral involving $\cos(A_i t_i)$ (Theorem 2.1). We distinguish between the cases n even and n odd. While it is true that once you see the formula ((2.1) or (2.2)), you may verify it directly (and this is the method used for the case of n even), we prefer to use for the case n odd a method which shows how the trigonometric-hyperbolic case is derived from the elementary exponential (parabolic) case $\exp\left(-\rho(A_1^2+\cdots+A_n^2)\right)$ through "inversion" of a transmutation formula [4]. The case n even could also be reduced to the case n odd via a descent argument. Note that (2.1) and (2.2) appear to be new even for the case of numbers (scalar operators). The cases $n=2,3$ are rather simple:

$$\cos\left(\sqrt{A^2+B^2}\,t\right)=\frac{1}{2\pi}\frac{\partial}{\partial t}\left[t\int_{\omega_1^2+\omega_2^2}\frac{\cos(t\omega_1 A)\cos(t\omega_2 B)}{\sqrt{1-(\omega_1^2+\omega_2^2)}}d\omega_1 d\omega_2\right] \tag{1.1}$$

$$\cos\left(\sqrt{A^2+B^2+C^2}\,t\right)=\frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\left[t\int_{S^2}\cos(t\omega_1 A)\cos(t\omega_2 B)\cos(t\omega_3 C)d\omega\right]. \tag{1.2}$$

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As a simple illustration, we derive the well-known formulas for the solution of the initial value problem for the wave equation in R^n from the elementary solution for the one-dimensional case (Example 4.1). Note that we do not use explicitly the spherical symmetry of Δ. We also solve the initial value problem for the Klein-Gordon equation.

In Section 3 we derive formulas for $\cos\left(\sqrt{A^2+B^2}\,t\right)$ when A and B do not commute. That the expression should involve a limit follows from the Trotter product formula for $e^{-(A^2+B^2)}$. Formal "inversion" of the Trotter formula leads to a limit formula for $\cos\left(\sqrt{A^2+B^2}\,t\right)$. This formula may be justified weakly (Proposition 3.1) or strongly on appropriate analytic vectors [5] (Theorem 3.2). One could express $\cos\left(\sqrt{A_1^2+\cdots+A_n^2}\,t\right)$ for general n (Remark 3.4) but the resulting formulas are quite heavy. The formulas (such as (3.4)) are reminiscent of those appearing in path integrals (such as the Feynman-Kac formulas for parabolic or Schrödinger equations). This similarity consists of the integrations being extended over balls of increasing dimensions; due to the hyperbolic nature of the problem, we have here in addition also differentiations to high orders.

To illustrate the formula for the non-commutative case we compute the propagators of the harmonic oscillator

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - x^2 u$$

(Example 4.2) and of simple hypoelliptic sum of squares operators, such as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + x_1^2 \frac{\partial^2 u}{\partial x_2^2}$$

and

$$\frac{\partial^2 u}{\partial t^2} = \Delta_H u$$

where Δ_H is the Laplacian on the Heisenberg group (Example 4.3). The expressions obtained are not very explicit.

Note that the operator $\frac{\sin\left(\sqrt{A^2+B^2}\,t\right)}{\sqrt{A^2+B^2}}$, and more generally, the operator $\frac{\sin\left(\sqrt{A_1^2+\cdots+A_n^2}\,t\right)}{\sqrt{A_1^2+\cdots+A_n^2}}$ may be represented by similar formulas, the only difference being dropping the left-most $\frac{\partial}{\partial t}$ from the corresponding formulas for $\cos\left(\sqrt{A^2+B^2}\,t\right)$ $\left(\cos\left(t\sqrt{A_1^2+\cdots+A_n^2}\right)\right)$. The proofs are similar to those of (2.2) or of Theorem 3.1.

I am very much indebted to V. Katsnelson for simulating discussions.

2 The commutative case

Theorem 2.1 *Let H be a Hilbert space, and let the (possibly unbounded) self-adjoint operators $A_1, \ldots, A_n$ commute.*

(i) *If n is odd, $n = 2m+1$, then*

$$\cos\left(t\sqrt{A_1^2+\cdots+A_n^2}\right) = \frac{1}{2(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Big[t^{2m-1} \times \int_{S^{2m}} \cos(t\omega_1 A_1)\cdots(t\omega_n A_n)d\omega\Big] \tag{2.1}$$

(here $d\omega$ denotes the surface measure on S^{2m}).
(ii) *If n is even, $n = 2m$, then*

$$\cos\left(t\sqrt{A_1^2+\cdots+A_n^2}\right) = \frac{1}{(2\pi)}^m\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Big[t^{2m-1} \times \int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1} \frac{\cos(t\omega_1 A_1)\cdots\cos(t\omega_{2m}A_{2m})}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}\, d\omega_1\cdots d\omega_{2m}\Big]\ . \tag{2.2}$$

Proof. Set $L = \sqrt{\sum_{i=1}^n A_i^2}$. Note that L is a non-negative self-adjoint operator in H. To prove (i), note that by commutativity we have for $\rho > 0$ that

$$e^{-L^2\rho} = \prod_{i=1}^n e^{-A_i^2\rho}\ . \tag{2.3}$$

The well-known transmutation formula (e.g. [4])

$$e^{-B^2\rho} = \frac{1}{\sqrt{4\pi\rho}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{4\rho}}\cos(Bt)dt\ , \tag{2.4}$$

valid for B self-adjoint, implies that

$$e^{-B^2\rho} = \frac{1}{\sqrt{\pi\rho}}\int_0^{\infty} e^{-\frac{t^2}{4\rho}}\cos(Bt)dt\ , \tag{2.4'}$$

Applying (2.4) to each A_i, we find that

$$\begin{aligned}\prod_{i=1}^n e^{-A_i^2\rho} &= \frac{1}{(4\pi\rho)^{n/2}}\int_{-\infty}^{\infty} e^{-\frac{t_1^2}{4\rho}}\cos(A_1t_1)dt_1\int_{-\infty}^{\infty} e^{-\frac{t_2^2}{4\rho}}\cos(A_2t_2)dt_2\cdots \\ &\quad\times\int_{-\infty}^{\infty} e^{-\frac{t_n^2}{4\rho}}\cos(A_nt_n)dt_n \\ &= \frac{1}{(4\pi\rho)^{n/2}}\int_{R^n} e^{-\frac{t_1^2+\cdots+t_n^2}{4\rho}}\cos(A_1t_1)\cdots\cos(A_nt_n)dt \\ &= \frac{1}{(4\pi\rho)^{n/2}}\int_0^{\infty} t^{n-1}e^{-\frac{t^2}{4\rho}}\int_{S^{n-1}}\cos(t\omega_1A_1)\cdots\cos(t\omega_nA_n)d\omega dt\ .\end{aligned} \tag{2.5}$$

Setting $B = L$ in (2.4′) we see from (2.3) and (2.5) that

$$\frac{1}{\sqrt{\pi\rho}}\int_0^\infty e^{-\frac{t^2}{4\rho}}\cos(Lt)dt = \frac{1}{(4\pi\rho)^{n/2}}\int_0^\infty t^{n-1}e^{-\frac{t^2}{4\rho}} \times \int_{S^{n-1}} \cos(t\omega_1 A_1)\cdots\cos(t\omega_n A_n)d\omega dt$$

so that

$$\int_0^\infty e^{-\frac{t^2}{4\rho}}\cos(Lt)dt = \frac{1}{2(4\pi\rho)^m}\int_0^\infty e^{-\frac{t^2}{4\rho}}t^{2m}F(t;A_1,\ldots,A_n)dt \tag{2.6}$$

where

$$F(t;A_1,\ldots,A_n) = \int_{S^{2m}} \cos(t\omega_1 A_1)\cos(t\omega_2 A_2)\cdots\cos(t\omega_n A_n)d\omega \ .$$

We apply successive integrations by parts to eliminate the factor ρ^{-m} that appears in the right hand side of (2.6). Thus,

$$\begin{aligned}
&\frac{1}{(4\pi\rho)^m}\int_0^\infty e^{-\frac{t^2}{4\rho}}t^{2m}F(t;A_1,\ldots,A_n)dt \\
&\quad= -\frac{1}{2\pi}\frac{1}{(4\pi\rho)^{m-1}}\int_0^\infty \frac{\partial}{\partial t}\left(e^{-\frac{t^2}{4\rho}}\right)t^{2m-1}F(t;A_1,\ldots,A_n)dt \\
&\quad= \frac{1}{2\pi}\frac{1}{(4\pi\rho)^{m-1}}\int_0^\infty e^{-\frac{t^2}{4\rho}}\frac{\partial}{\partial t}\left[t^{2m-1}F(t;A_1,\ldots,A_n)\right]dt \\
&\quad= -\frac{1}{(2\pi)^2}\frac{1}{(4\pi\rho)^{m-2}}\int_0^\infty \left(\frac{1}{t}\frac{\partial}{\partial t}\right)\left(e^{-\frac{t^2}{4\rho}}\right)\frac{\partial}{\partial t}\left[t^{2m-1}F(t;A_1,\ldots,A_n)\right]dt \\
&\quad= \frac{1}{(2\pi)^2}\frac{1}{(4\pi\rho)^{m-2}}\int_0^\infty e^{-\frac{t^2}{4\rho}}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)\right]\left(t^{2m-1}F(t;A_1,\ldots,A_n)\right]dt \\
&\quad= \cdots = \frac{1}{(2\pi)^m}\int_0^\infty e^{-\frac{t^2}{4\rho}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[t^{2m-1}F(t;A_1,\ldots,A_n)\right]dt
\end{aligned}$$

Substituting in (2.6) and applying the uniqueness theorem for the Laplace transform, we obtain (2.1).

One could derive (2.2) from (2.1) by setting $A_{2m+1} = 0$ (essentially a method of descent). We prefer an alternative method. Let $a_1,\ldots,a_{2m}$ be scalars. Then (using multi-index notation)

$$\begin{aligned}
&\int_{\|\omega\|\le 1}\frac{\cos(\omega_1 a_1 t)\cdots\cos(\omega_{2m}a_{2m}t)}{\sqrt{1-\parallel\omega\parallel^2}}d\omega_1\cdots d\omega_{2m} \\
&\qquad= \sum_\alpha (-1)^\alpha\frac{t^{2|\alpha|}a^{2\alpha}}{(2\alpha)!}\int_{\|\omega\|\le 1}\frac{\omega^{2\alpha}}{\sqrt{1-\parallel\omega\parallel^2}}d\omega_1\cdots d\omega_{2m} \ .
\end{aligned}\tag{2.7}$$

The integral on the right hand side of (2.7) is essentially the Dirichlet integral [8]. In fact,

$$\int_{\|\omega\|^2\leq 1} \frac{\omega^{2\alpha}}{\sqrt{1-\parallel\omega\parallel^2}} d\omega_1\cdots d\omega_{2m} = 2^{2m}\int_{\omega\geq 0,\, \|\omega\|^2\leq 1} \frac{w^{2\alpha}}{\sqrt{1-\parallel\omega\parallel^2}} d\omega_1\cdots d\omega_n \ .$$

Set $z_i=\omega_i^2$, $1\leq i\leq 2m$. Then the integral on the right hand side of (2.7) is equal to

$$\int_{z\geq 0,\, z_1+\cdots z_{2m}\leq 1} z_1^{\alpha_1-\frac{1}{2}}\cdots z_{2m}^{\alpha_{2m}-\frac{1}{2}}(1-z_1-\ldots-z_{2m})^{-1/2}dz_1\cdots dz_{2m} \ .$$

According to formulas (7.7.4) and (7.7.5) in [8], the value of this integral is

$$\frac{\Gamma\left(\alpha_1+\frac{1}{2}\right)\cdots\Gamma\left(\alpha_{2m}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(|\alpha|+m+\frac{1}{2}\right)} \ .$$

Substituting in (2.7) and recalling the duplication formula for the Gamma function

$$\Gamma\left(k+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2k)}{2^{2k-1}\Gamma(k)}$$

we see that

$$\begin{aligned}
&\int_{\|\omega\|^2\leq 1} \frac{\cos(\omega_1 a_1 t)\cdots\cos(\omega_{2m}a_{2m}t)}{\sqrt{1-\parallel\omega\parallel^2}} d\omega_1\cdots d\omega_{2m}\\
&=\sum_\alpha (-1)^{|\alpha|}\frac{t^{2|\alpha|}a^{2\alpha}}{(2\alpha)!}\frac{\Gamma\left(\alpha_1+\frac{1}{2}\right)\cdots\Gamma\left(\alpha_{2m}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(|\alpha|+m+\frac{1}{2}\right)}\\
&=\sum_\alpha \frac{(-1)^{|\alpha|}}{(2\alpha)!}\frac{t^{2|\alpha|}a^{2\alpha}\pi^m}{2^{2|\alpha|-2m}}\frac{\Gamma(2\alpha_1)\cdots\Gamma(2\alpha_{2m})2^{2|\alpha|+2m-1}\Gamma(|\alpha|+m)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_{2m})\Gamma(2|\alpha|+2m)}\\
&=\sum_\alpha \frac{(-1)^{|\alpha|}t^{2|\alpha|}a^{2\alpha}\pi^m 2^{4m}}{(2\alpha_1)\cdots(2\alpha_{2m})\Gamma(\alpha_1)\cdots\Gamma(\alpha_{2m})}\frac{1}{2^{|\alpha|+m}(2m+2|\alpha|-1)}\\
&\qquad\times\frac{1}{(2m+2|\alpha|-3)\cdots 1}\\
&=\sum_\alpha \frac{(-1)^{|\alpha|}t^{2|\alpha|}a^{2\alpha}}{\alpha!}\frac{\pi^m 2^{4m}}{2^{2m}}\frac{1}{2^{|\alpha|+m}}\frac{1}{(2m+2|\alpha|-1)(2m+2|\alpha|-3)\cdots 1}\\
&=\sum_\alpha \frac{(-1)^{|\alpha|}t^{2|\alpha|}a^{2\alpha}(2\pi)^m}{2^{|\alpha|}\alpha!}\frac{1}{(2m+2|\alpha|-1)(2m+2|\alpha|-3)\cdots 1} \ .
\end{aligned}$$

Hence

$$t^{2m-1} \int_{\|\omega\|^2 \le 1} \frac{\cos(\omega_1 a_1 t) \cdots \cos(\omega_{2m} a_{2m} t)}{\sqrt{1 - \| \omega \|^2}} d\omega_1 \cdots d\omega_{2m}$$
$$= \sum_\alpha \frac{(-1)^{|\alpha|} (2\pi)^m}{2^{|\alpha|} \alpha!} \frac{t^{2|\alpha|+2m-1} a^{2\alpha}}{(2|\alpha| + 2m - 1)(2|\alpha| + 2m - 3) \cdots 1} \; . \tag{2.8}$$

But

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right) t^{2|\alpha|+2m-1} = (2|\alpha| + 2m - 1) t^{2|\alpha|+2m-3}, \cdots, \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} t^{2|\alpha|+2m-1}$$
$$= (2|\alpha| + 2m - 1) \cdots (2|\alpha| + 3) t^{2|\alpha|+1} \; .$$

Applying to (2.8), we get

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \left[t^{2m-1} \int_{\|\omega\|^2 \le 1} \frac{\cos(\omega_1 a_1 t) \cdots \cos(\omega_{2m} a_{2m} t)}{\sqrt{1 - \| \omega \|^2}} d\omega_1 \cdots d\omega_{2m} \right]$$
$$= \sum_\alpha \frac{(-1)^{|\alpha|} (2\pi)^m a^{2\alpha}}{2^{|\alpha|} \alpha!} \frac{t^{2|\alpha|+1}}{(2|\alpha| + 1)(2|\alpha| - 1) \cdots 1}$$

and

$$\frac{1}{(2\pi)^m} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \left[t^{2m-1} \int_{\|\omega\|^2 \le 1} \frac{\cos(\omega_1 a_1 t) \cdots \cos(\omega_{2m} a_{2m} t)}{\sqrt{1 - \| \omega \|^2}} d\omega_1 \cdots d\omega_{2m} \right]$$
$$= \sum_\alpha \frac{(-1)^{|\alpha|} a^{2\alpha} t^{2|\alpha|}}{2^{|\alpha|} \alpha! (2|\alpha| - 1) \cdots 1} \; . \tag{2.9}$$

On the other hand, for every non-negative integer k,

$$\sum_{|\alpha|=k} \frac{a^{2\alpha}}{\alpha!} = \frac{1}{k!} \sum_{|\alpha|=k} \frac{k!}{\alpha!} a^{2\alpha} = \frac{1}{k!} (a_1^2 + \cdots + a_{2m}^2)^k \; .$$

Inserting in (2.9) we conclude that

$$\frac{1}{(2\pi)^m} \frac{\partial}{\partial t} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \left[t^{2m-1} \int_{\|\omega\|^2 \le 1} \frac{\cos(\omega_1 a_1 t) \cdots \cos(\omega_{2m} a_{2m} t)}{\sqrt{1 - \| \omega \|^2}} d\omega_1 \cdots d\omega_{2m} \right]$$
$$= \sum_{k=0}^\infty \frac{(-1)^k}{2^k k!} \frac{t^{2k} (a_1^2 + \cdots + a_{2m}^2)^k}{(2k-1) \cdots} = \sum_{k=1}^\infty \frac{(-1)^k (|a| t)^{2k}}{(2k)!} = \cos(|a| t)$$

proving (2.2) for the case $A_i = a_i$, $i = 1, \ldots, 2m$. In general, let $dE^{(i)}(\lambda)$ denote the spectral measure of A_i, $1 \le i \le 2m$. The commutativity assumption implies

that

$$\begin{aligned}
&\cos\left(t\sqrt{A_1^2+\cdots+A_{2m}^2}\right) \\
&\qquad = \int \cos\left(t\sqrt{\lambda_1^2+\cdots\lambda_{2m}^2}\right) dE^{(1)}(\lambda_1)\cdots dE^{(2m)}(\lambda_{2m}) \ .
\end{aligned} \tag{2.10}$$

Application of (2.2) (case of real numbers) to (2.10) implies that

$$\begin{aligned}
&\cos\left(t\sqrt{A_1^2+\cdots+A_{2m}^2}\right) = \int \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{2m-1}\Bigg[t^{2m-1} \\
&\qquad \times \int_{\|\omega\|^2\le 1} \frac{\cos(\omega_1\lambda_1 t)\cdots\cos(\omega_{2m}\lambda_{2m}t)}{\sqrt{1-\parallel\omega\parallel^2}} d\omega_1\cdots d\omega_{2m}\Bigg] \\
&\qquad \times dE^{(1)}(\lambda_1)\cdots dE^{(2m)}(\lambda_{2m}) \\
&= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{2m-1}\Bigg\{t^{2m-1} \\
&\qquad \times \int_{\|\omega\|^2\le 1}\Bigg[\int \frac{\cos(\omega_1\lambda_1 t)\cdots\cos(\omega_{2m}\lambda_{2m}t)}{\sqrt{1-\parallel\omega\parallel^2}} \\
&\qquad \times dE^{(1)}(\lambda_1)\cdots dE^{(2m)}(\lambda_{2m})\Bigg] d\omega_1\cdots d\omega_{2m}\Bigg\} \\
&= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{2m-1}\left[t^{2m-1}\int \frac{\cos(\omega_1 A_1 t)\cdots\cos(\omega_{2m}A_{2m}t)}{\sqrt{1-\parallel\omega\parallel^2}} d\omega_1\cdots d\omega_{2m}\right]
\end{aligned}$$

proving (2.2). ∎

3 The non-commutative case

Let A, B be (not necessarily commuting) self-adjoint operators on the Hilbert space H. Then the operators A^2, B^2 are semi-bounded. Recall the Trotter product formula [6]

$$e^{-\rho(A^2+B^2)} = \lim_{m\to\infty}\left[e^{-\rho\frac{A^2}{m}}e^{-\rho\frac{B^2}{m}}\right]^m \tag{3.1}$$

valid for $\rho > 0$ (or, more generally, for $\operatorname{Re}\rho \ge 0$). Setting $n = 2m+1$,

$$\begin{aligned}
&A_1 = \frac{A}{\sqrt{m}}\ ,\ A_2 = \frac{B}{\sqrt{m}}\ ,\ A_3 = \frac{A}{\sqrt{m}}\ ,\ \ldots\ ,\ A_{2m-1} = \frac{A}{\sqrt{m}}\ ,\ A_{2m} = \frac{B}{\sqrt{m}}\ , \\
&A_{2m+1} = 0\ ,
\end{aligned}$$

we infer from (2.5) that

$$\left[e^{-\rho\frac{A^2}{m}}e^{-\rho\frac{B^2}{m}}\right]^m = \frac{1}{(4\pi\rho)^{m+1/2}}\int_0^\infty t^{n-1}e^{-\frac{t^2}{4\rho}}\int_{S^{2m}}\cos\left(\frac{t\omega_1 A}{\sqrt{m}}\right)$$
$$\times\cos\left(\frac{t\omega_2 B}{\sqrt{m}}\right)\cdots\cos\left(\frac{t\omega_{2m-1}A}{\sqrt{m}}\right)\cos\left(\frac{t\omega_{2m}B}{\sqrt{m}}\right)d\omega dt\ . \tag{3.2}$$

Note that $d\omega = \frac{2d\omega_1\cdots d\omega_{2m}}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}$. Integrating by parts as in Section 2, we see from (3.1) and (3.2) that

$$\int_0^\infty e^{-\frac{t^2}{4\rho}}\cos\left(\sqrt{A^2+B^2}\,t\right)dt = \lim_{m\to\infty}\int_0 e^{-\frac{t^2}{4\rho}}\frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}$$
$$\times\int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}$$
$$\times\frac{\cos\left(\frac{t\omega_1 A}{\sqrt{m}}\right)\cos\left(\frac{t\omega_2 B}{\sqrt{m}}\right)\cdots\cos\left(\frac{t\omega_{2m-1}A}{\sqrt{m}}\right)\cos\left(\frac{t\omega_{2m}B}{\sqrt{m}}\right)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}d\omega_1\cdots d\omega_{2m}\Bigg]dt\ . \tag{3.3}$$

The relation (3.3) suggests, at least formally, that

$$\cos\left(\sqrt{A^2+B^2}\,t\right) = \lim_{m\to\infty}\frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}\int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}$$
$$\times\frac{\cos\left(\frac{t\omega_1 A}{\sqrt{m}}\right)\cos\left(\frac{t\omega_2 B}{\sqrt{m}}\right)\cdots\cos\left(\frac{t\omega_{2m-1}A}{\sqrt{m}}\right)\cos\left(\frac{t\omega_{2m}B}{\sqrt{m}}\right)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}d\omega_1\cdots d\omega_{2m}\Bigg]\ . \tag{3.4}$$

In the rest of this section we try to interpret and to justify the heuristically obtained formula (3.4). Note that for each m, the function

$$C_m(t) = \int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}\cos\left(\frac{t\omega_1 A}{\sqrt{m}}\right)\cdots\cos\left(\frac{t\omega_{2m}B}{\sqrt{m}}\right)d\omega_1\cdots d\omega_{2m}$$

is a holomorphic function from the complex numbers to the space of bounded operators on H. Thus we "only" have to make precise the convergence in (3.4). Our first result is about weak convergence.

Proposition 3.1 *Let A^2+B^2 be essentially self-adjoint on $D(A^2)\cap D(B^2)$. For every positive integer m and $h\in H$, define the function $F_m(t;A,B;h)\in C(R^1;H)$ by*

$$F_m(t;A,B;h) = \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}$$
$$\times\int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}\frac{\cos\left(\frac{\omega_1 tA}{\sqrt{m}}\right)\cdots\cos\left(\frac{\omega_{2m}tB}{\sqrt{m}}\right)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}d\omega_1\cdots d\omega_{2m}h\Bigg]\ . \tag{3.5}$$

Then the functions $F_m(t;A,B;h)$ converge as vector valued temperate distributions (i.e., in $S'(R^1;H)$) to $\cos\left(\sqrt{A^2+B^2}\,t\right)h$.

Proof. Let $\operatorname{Re}\rho \geq 0$, set $\frac{1}{2\rho} = i\sigma$ so that $\operatorname{Im}\sigma \leq 0$, and let $\frac{t^2}{2} = s$. It follows from (3.2) and (3.3) that

$$\left[e^{-\frac{A^2}{2i\sigma m}}e^{-\frac{B^2}{2i\sigma m}}\right]^m h = \sqrt{\frac{2i\sigma}{\pi}}\int_0^\infty F_m\left(\sqrt{2s}\,;A,B;h\right)e^{-is\sigma}\frac{ds}{\sqrt{2s}} \tag{3.6}$$

and, similarly,

$$e^{-\frac{A^2+B^2}{2i\sigma}}h = \sqrt{\frac{2i\sigma}{\pi}}\int_0^\infty \cos\left(\sqrt{A^2+B^2}\,\sqrt{2s}\right)he^{-is\sigma}\frac{ds}{\sqrt{2s}} \tag{3.7}$$

when $\sqrt{2i\sigma}$ is chosen to be positive for σ on the negative imaginary axis. The left hand sides of (3.6) and (3.7) are uniformly bounded (by $\| h \|$) for $\operatorname{Im}\sigma \geq 0$, and are holomorphic for $\operatorname{Im}\sigma > 0$. Hence $\sqrt{\frac{\pi}{2i\sigma}}\left[e^{-\frac{A^2}{2i\sigma m}}e^{-\frac{B^2}{2i\sigma m}}\right]^m h$ is the Fourier transform of $F_m\left(\sqrt{2s}\,;A,B;h\right)/\sqrt{2s}$ and $\sqrt{\frac{\pi}{2i\sigma}}e^{-\frac{A^2+B^2}{2i\sigma}}h$ is the Fourier transform of $\frac{\cos\left(\sqrt{A^2+B^2}\,\sqrt{2s}\right)}{\sqrt{2s}}$. By the Trotter product formula, the left hand sides of (3.6) converge to the left side of (3.7) for every $\sigma \in R^1\backslash\{0\}$. Let $\psi(\sigma) \in S(R^2,H)$ be arbitrary. By the dominated convergence theorem

$$\int\left\langle\left(e^{-\frac{A^2}{2i\sigma m}}e^{-\frac{B^2}{2i\sigma m}}\right)^m h,\psi(\sigma)\right\rangle\frac{d\pi}{\sqrt{\sigma}} \to \int\left\langle e^{-\frac{A^2+B^2}{2i\sigma}}h,\psi(\sigma)\right\rangle\frac{d\sigma}{\sqrt{\sigma}}\ .$$

The continuity of the inverse Fourier transform in $S'(R^1,H)$ implies that the sequence $F_m\left(\sqrt{2s}\,;A,B;h\right)/\sqrt{2s}$ converges, in the temperate distribution sense, to $\cos\left(\sqrt{A^2+B^2}\,\sqrt{2s}\right)\sqrt{2s}$. Passing back to $t = \sqrt{2s}$ and noting that $F_m(t;A,B;h)$ and $\cos\left(\sqrt{A^2+B^2}\,t\right)$ are even functions, we obtain the result. ∎

One could possibly get a more precise version of Proposition 3.1 by using a strong version of the Trotter product formula and by estimating the convergence in an appropriate Sobolev norm. We state now a result on pointwise convergence, under stronger assumptions on h.

Theorem 3.1 *Let $h \in H$ be such that there exist constants C, K so that for every multi-index α with $2m$ components $h \in D\left(A^{\alpha_1}B^{\alpha_2}\cdots A^{\alpha_{2n-1}}B^{\alpha_{2n}}\right))$ and*

$$\| A^{\alpha_1}B^{\alpha_2}\cdots A^{\alpha_{2n-1}}B^{\alpha_{2n}}h \| \leq CK^{|\alpha|}|\alpha|!\ . \tag{3.8}$$

Then $F_m(t;A,B;h) \to \cos\left(\sqrt{A^2+B^2}\,t\right)h$ for $|t| < \frac{1}{\sqrt{2}K}$, uniformly in compact subintervals of $\left(-\frac{1}{\sqrt{2}K},\frac{1}{\sqrt{2}K}\right)$.

Proof. Note that for every positive integer n,

$$(A^2+B^2)^n = \sum_{\varepsilon_i=0,1,\ 1\leq i\leq n} A^{2\varepsilon_1}B^{2(1-\varepsilon_1)}\cdots A^{2\varepsilon_n}B^{2(1-\varepsilon_n)}\ . \tag{3.9}$$

By (3.8) we have for h satisfying the assumptions that

$$\| (A^2+B^2)^n h \| \leq 2^n C K^{2n} (2n)! \tag{3.10}$$

Expanding $\cos\left(\sqrt{A^2+B^2}\,t\right) h$ in Taylor series, we see that for every N,

$$\| \cos\left(\sqrt{A^2+B^2}\,t\right) h - \sum_{n\leq N} \frac{(-1)^n}{(2n)!} t^{2n} (A^2+B^2)^n h \|$$

$$\leq \sum_{n=N+1} \frac{|t|^{2n}}{(2n)!} 2^n C K^{2n} (2n)! = C \sum_{n=N+1}^{\infty} \left(\sqrt{2}\,|t| K\right)^{2n} = \frac{C\left(\sqrt{2}\,tK\right)^{2N+2}}{1-2t^2K^2} \tag{3.11}$$

or that the series converges absolutely and uniformly in compact subsets of $|t| < \frac{1}{\sqrt{2}\,K}$.

Expanding similarly $\cos\left(\frac{\omega_1 tA}{\sqrt{m}}\right)\cos\left(\frac{\omega_2 tB}{\sqrt{m}}\right)\cdots\cos\left(\frac{\omega_{2m} tB}{\sqrt{m}}\right) h$ in Taylor series, we get the estimate

$$\Big| \cos\left(\frac{\omega_1 tA}{\sqrt{m}}\right)\cos\left(\frac{\omega_2 tB}{\sqrt{m}}\right)\cdots\cos\left(\frac{\omega_{2m-1} tA}{\sqrt{m}}\right)\cos\left(\frac{\omega_{2m} tB}{\sqrt{m}}\right) h - \sum_{n=0}^{N} (-1)^n$$

$$\times \frac{t^{2n}}{m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{(\omega_1 A)^{2\alpha_1}(\omega_2 B)^{2\alpha_2}\cdots(\omega_{2n-1} A)^{2\alpha_{2m-1}}(\omega_{2m} B)^{2\alpha_{2m}} h}{(2\alpha)!} \Big|$$

$$\leq \sum_{n=N+1}^{\infty} \frac{|t|^{2n}}{m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{\| A^{2\alpha_1} B^{2\alpha_2} \cdots A^{2\alpha_{2m-1}} B^{2\alpha_{2m}} h \|}{(2\alpha)!}$$

$$\leq \sum_{n=N+1}^{\infty} \frac{t^{2n}}{m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{CK^{2n}(2n)!}{(2\alpha)!} \leq C \sum_{n=N+1}^{\infty} (tK)^{2n} \frac{(2m)^{2n}}{m^n}$$

$$= C \sum_{n=N+1}^{\infty} (4t^2K^2m)^n$$

so that this expansion converges absolutely and uniformly in the product of $\| \omega \| \leq 1$ with compact subsets of $|t| < \frac{1}{2K\sqrt{m}}$; $\frac{1}{2K\sqrt{m}}$ is positive for every *fixed* m. Hence we may perform the integration in (3.5) term by term. Evaluating the integral $\int_{\|\omega\|\leq 1} \frac{\omega^{2\alpha}}{\sqrt{1-\|\omega\|^2}} d\omega_1 \cdots d\omega_{2m}$ as in Section 2, we obtain the non-commutative version of (2.9) with $a_i = \frac{A}{\sqrt{m}}$ for i even, $a_i = \frac{B}{\sqrt{m}}$ for i odd, i.e.,

that

$$F_m(t;A,B;h) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \frac{t^{2n}}{m^n(2n-1)\cdots 1} \times \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{A^{2\alpha_1}B^{2\alpha_2}\cdots A^{2\alpha_{2m-1}}B^{2\alpha_{2m}}h}{\alpha!} , \tag{3.12}$$

the series converging absolutely and uniformly in compact subsets of $|t| < \frac{1}{2K\sqrt{m}}$. Noting that $2^n n!(2n-1)\cdots 1 = (2n)!$, we may write (3.12) in the form

$$F_m(t;A,B;h) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{n!}{\alpha!} A^{2\alpha_1}B_{2\alpha_2}\cdots A^{2\alpha_{2m-1}}B^{2\alpha_m}h . \tag{3.13}$$

The n^{th} term (sum) in (3.11) is majorized according to the assumption (3.8) by

$$\frac{|t|^{2n}}{(2n)!m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{n!}{\alpha!} CK^{2n}(2n)!$$

$$= C\frac{(|t|^2K^2)^n}{m^n} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{n!}{\alpha!} = \frac{C(|t|^2K^2)^n}{m^n}(2m)^n = C(2|t|^2K^2)^n$$

so that the radius of convergence of the series in (3.11) is $\frac{1}{\sqrt{2}\,K}$, and by analyticity of $F_m(t;A,B;h)$ (as a function of t) the series expansion (3.13) is valid for $|t| < \frac{1}{\sqrt{2}\,K}$, *uniformly in m.*

Let X, Y be non-commuting finite dimensional matrices. In this case the convergence in the Trotter product formula [6]

$$e^{z(X+Y)} = \lim_{n\to\infty} \left(e^{zX/m}e^{zY/m}\right)^m$$

is uniform in compact subset of the complex plane. Hence the Taylor coefficients of the right hand side converge to those of the left hand side, or

$$\frac{(X+Y)^n}{n!} = \lim_{m\to\infty} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{X^{\alpha_1}Y^{\alpha_2}\cdots X^{\alpha_{2n-1}}Y^{\alpha_{2n}}}{m^n\alpha!} \tag{3.14}$$

for every (fixed) n. It follows from (3.8) that for every (fixed) n and h,

$$\frac{(A^2+B^2)^n}{n!}h = \lim_{m\to\infty} \sum_{\substack{\alpha\in Z_+^{2m} \\ |\alpha|=n}} \frac{A^{2\alpha_1}B^{2\alpha_2}\cdots A^{2\alpha_{2m-1}}B^{\alpha_m}}{m^n\alpha!}h . \tag{3.15}$$

For any given $\varepsilon = 0$ and $T < \frac{1}{\sqrt{2}\,K}$, choose N such that the right hand side of (3.11) and

$$\| \, F_m(t;A,B;h) - \sum_{n=0}^{N} \frac{(-1)^n}{(2n)!} \frac{t^{2n}}{m^n} \sum_{\substack{\alpha \in Z_+^{2m} \\ |\alpha|=n}} \frac{n!}{\alpha!} A^{2\alpha_1} B_{2\alpha_2} \cdots B^{2\alpha_{2m}} h \, \|$$

are each less than $\frac{\varepsilon}{3}$ for $|t| \leq T$. By (3.15) there exists a constant $M > 0$ such that for every $0 \leq n \leq N$ we have

$$\| \, \frac{t^{2n}}{(2n)!m^n} \sum_{\substack{\alpha \in Z_+^{2m} \\ |\alpha|=n}} \frac{n!A^{2\alpha_1} B^{2\alpha_2} \cdots B^{2\alpha_{2m}}}{\alpha!} h - \frac{t^{2n}}{(2n)!} (A^2 + B^2)^n h \, \| < \frac{\varepsilon}{3(N+1)}$$

for $|t| \leq T$ and $m > M$. Hence

$$\| \, F_m(t;A,B;h) - \cos\left(\sqrt{A^2+B^2}\, t\right) h \, \| < \varepsilon$$

for $|t| \leq T$ and $m > M$. ∎

Remark 3.3 If for every $K > 0$ there exists a $C > 0$ such that (3.8) holds for all α then $F_m(t;A,B;h) \to \cos\left(\sqrt{A^2+B^2}\, t\right) h$ uniformly in compact subsets of the complex t plane. In particular, if A and B are bounded, then this holds for every h in H (uniformly in bounded sets of H).

Remark 3.4 The Trotter product formula for any number of non-commuting operators is well-known [7]. In analogy to (3.4), we may obtain the formula

$$\begin{aligned} \cos\left(\sqrt{A_1^2 + \cdots + A_q^2}\, t\right) &= \lim_{m\to\infty} \frac{1}{(2\pi)^{mq/2}} \frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{mq}{2}-1} \Bigg[t^{mq-1} \\ &\times \int_{\omega_1^2 + \cdots + \omega_{mq}^2 \leq 1} \cos\left(\frac{t\omega_1 A_1}{\sqrt{m}}\right) \cdots \cos\left(\frac{t\omega_q A_q}{\sqrt{m}}\right) \cos\left(\frac{t\omega_{q+1} A_1}{\sqrt{m}}\right) \cdots \\ &\times \cos\left(\frac{t\omega_{2q} A_q}{\sqrt{m}}\right) \frac{\cos\left(\frac{t\omega_{(m-1)q+1} A_1}{\sqrt{m}}\right) \cdots \cos\left(\frac{t\omega_{mq} A_q}{\sqrt{m}}\right)}{\sqrt{1 - (\omega_1^2 + \cdots + \omega_{mq}^2)}} d\omega_1 \cdots d\omega_{mq} \Bigg] \end{aligned} \tag{3.16}$$

where m run over all positive integers if q is even, and over even m only if q is odd. (Note that if mq is odd then we may replace the right hand side of (3.16) by a term involving integration over S^{mq}, as in (2.1).) The formal limit (3.16) may be interpreted, under suitable assumptions, as in Proposition 3.1 or in Theorem 3.2. We leave the details to the diligent reader.

Remark 3.5 Analytic domination has been applied in the context of Trotter products in [1].

4 Illustrations and examples

We observe that in all ω-integrations appearing earlier we may replace a term of the form $\cos(\omega_j T)$ by $\exp(i\omega_j T)$ (or by $\exp(-i\omega_j T)$) since the rest of the integrand is even in ω_j and $\sin(\omega_j T)$ is odd. We will use this observation constantly in this section in order to simplify a few calculations.

Example 4.1 *The wave equation in R^n — the method of ascent.*
Recall that the operator $\frac{1}{i}\frac{d}{dx}$ has a (unique) self-adjoint realization in $L^2(R^1)$ and the distribution kernels of $\cos\left(\frac{t}{i}\frac{d}{dx}\right)$, $\exp\left(it\frac{1}{i}\frac{d}{dx}\right)$ and $\exp\left(-it\frac{1}{i}\frac{d}{dx}\right)$ (solutions of the one-dimensional wave equation) are given by

$$\begin{aligned}
\cos\left(\frac{t}{i}\frac{d}{dx}\right)(x,\overline{x}) &= \frac{\delta(x-\overline{x}+t)+\delta(x-\overline{x}-t)}{2} \\
\exp\left(it\frac{1}{i}\frac{d}{dx}\right)(x,\overline{x}) &= \delta(x-\overline{x}+t) \qquad (4.1)\\
\exp\left(-it\frac{1}{i}\frac{d}{dx}\right)(x,\overline{x}) &= \delta(x-\overline{x}-t)\ .
\end{aligned}$$

The simplest cases are $n = 2, 3$. In the two dimensional case we obtain from (1.1) or (2.2) ($m = 1$) that

$$\begin{aligned}
&\cos\left(t\sqrt{-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}}\right)(x,y;\overline{x},\overline{y}) \\
&\quad = \frac{1}{2\pi}\frac{\partial}{\partial t}\left[t\int_{\omega_1^2+\omega_2^2\le 1}\frac{\delta(x-\overline{x}+\omega_1 t)\delta(y-\overline{y}+\omega_2 t)}{\sqrt{1-\omega_1^2-\omega_2^2}}d\omega_1 d\omega_2\right]\ .
\end{aligned}$$

But

$$\begin{aligned}
&\int_{\omega_1^2+\omega_2^2\le 1}\frac{\delta(x-\overline{x}+\omega_1 t)\delta(y-\overline{y}+\omega_2 t)}{\sqrt{1-(\omega_1^2+\omega_2^2)}}d\omega_1 d\omega_2 \\
&\quad = \int_{\omega_1^2+\omega_2^2\le 1}\frac{\delta\left(\frac{x-\overline{x}}{t}+\omega_1\right)\delta\left(\frac{y-\overline{y}}{t}+\omega_2\right)}{t^2\sqrt{1-(\omega_1^2+\omega_2^2)}}d\omega_1 d\omega_2 \\
&\quad = \frac{1}{t^2}\frac{H\left(t^2-(x-\overline{x})^2-(y-\overline{y})^2\right)}{\sqrt{1-\left[\frac{(x-\overline{x})^2}{t^2}+\frac{(y-\overline{y})^2}{t^2}\right]}} = \frac{1}{t}\frac{H\left(t^2-(x-\overline{x})^2-(y-\overline{y})^2\right)}{\sqrt{t^2-(x-\overline{x})^2+(y-\overline{y})^2}}
\end{aligned}$$

where H is the Heaviside function. Hence

$$\cos\left(t\sqrt{-\left(\frac{\partial}{\partial x}\right)^2-\left(\frac{\partial}{\partial y}\right)^2}\right) = \frac{1}{2\pi}\frac{\partial}{\partial t}\frac{H\left(t^2-(x-\overline{x})^2-(y-\overline{y})^2\right)}{\sqrt{t^2-(x-\overline{x})^2+(y-\overline{y})^2}}\ . \qquad (4.2)$$

The three dimensional case is actually simpler:

$$\cos\left(t\sqrt{-\left(\frac{\partial}{\partial x}\right)^2-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}}\right)(x,y,z;\overline{x},\overline{y},\overline{z})$$
$$=\frac{1}{4\pi}\frac{\partial}{\partial t}\left[t\int_{S^2}\left(e^{-it\omega_1\frac{1}{i}\frac{\partial}{\partial x}}\right)(x,\overline{x})\left(e^{-it\omega_2\frac{1}{i}\frac{\partial}{\partial y}}\right)(y,\overline{y})\left(e^{-it\omega_3\frac{1}{i}\frac{\partial}{\partial z}}\right)(z,\overline{z})d\omega\right]$$
$$=\frac{1}{4\pi}\frac{\partial}{\partial t}\left[t\int_{S^2}\delta(x-\overline{x}-t\omega_1)\delta(y-\overline{y}-t\omega_2)\delta(z-\overline{z}-t\omega_3)d\omega\right]$$
$$=\frac{1}{4\pi}\frac{\partial}{\partial t}\left[t\int_{S^2}\delta\left(x-(\overline{x}+t\omega_1)\right)\delta\left(y-(\overline{y}+t\omega_2)\right)\delta\left(z-(\overline{z}+t\omega_3)\right)d\omega\right]. \tag{4.3}$$

The formulas (4.2) and (4.3) are classical.

More generally, for $n\geq 3$ odd, $n=2m+1$, we have from (2.1) that

$$\cos\left(t\sqrt{-\Delta_n}\right)=\cos\left(t\sqrt{-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}}\right)(x,\overline{x})$$
$$=\frac{1}{2(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}$$
$$\times\int_{S^{2m}}\left(e^{-i\omega_1 t\frac{1}{i}\frac{\partial}{\partial x_1}}\right)(x_1,\overline{x}_1)\cdots\left(e^{-i\omega_n t\frac{1}{i}\frac{\partial}{\partial x_n}}\right)(x_n,\overline{x}_n)d\omega\Bigg] \tag{4.4}$$
$$=\frac{1}{2(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}$$
$$\times\int_{S^{2m}}\delta(x_1-\overline{x}_1-\omega_1 t)\cdots\delta(x_n-\overline{x}_n-\omega_n t)d\omega\Bigg].$$

But

$$1\cdot 3\cdots(n-2)\omega_n=1\cdot 3\cdots(n-2)\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}=2(2\pi)^m \tag{4.5}$$

and we recapture the classical formula [2, 3]. For general even dimensions $n=2m$, we get from (2.2) that

$$\cos\left(t\sqrt{-\Delta_{2m}}\right)(x,\overline{x})=\frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}\int_{\omega_1^2+\cdots+\omega_{2m}^2\leq 1}$$
$$\times\frac{\delta(x_1-\overline{x}_1-\omega_1 t)\cdots\delta(x_{2m}-\overline{x}_{2m}-\omega_{2m}t)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}d\omega_1\cdots d\omega_{2m}\Bigg]. \tag{4.6}$$

Applying (4.5) (with $n=2m+1$) we see that the right hand side of (4.6) coincides with the well-known kernel of the operator mapping $u(x,0)$ to $u(x,t)$

(when $\frac{\partial u}{\partial t}(x,0)=0$) for even n [2, 3]. Thus, we are able to build the solution of the initial value problem for the n-dimensional wave equation from the (very elementary) one-dimensional solution.

The Klein-Gordon operator K is given by $K=\Delta-a^2$, and we wish to represent the solution of the initial value problem for the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2}=\Delta u-a^2 u \tag{4.7}$$

by calculating the distribution kernel of

$$\cos\left(\sqrt{-K}\,t\right)=\cos\left(\sqrt{\sum_{j=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_j}\right)^2+a^2}\,t\right)\ .$$

Set $A_j=\frac{1}{i}\frac{\partial}{\partial x_j}$, $j=1,\ldots,n$, $A_{n+1}=a$. If n is even, $n=2m$, we apply (2.1) to get

$$\begin{aligned}
\cos\left(\sqrt{-K}\,t\right)(x,\overline{x}) &= \frac{1}{2(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1} \\
&\quad\times\int_{S^{2m}}\cos(a\omega_{2m+1}t)\prod_{j=1}^{2m}\delta(x_j-\overline{x}_j-\omega_j t)d\omega\Bigg] \\
&= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg\{t^{2m-1}\int_{\omega_1^2+\cdots+\omega_{2m-1}^2\le 1}\frac{\cos\left(at\sqrt{1-\omega_1^2-\cdots-\omega_{2m}^2}\right)}{\sqrt{1-\omega_1^2-\cdots-\omega_{2m}^2}} \\
&\quad\times\prod_{j=1}^{2m}\left[\delta\left(\frac{x_j-\overline{x}_j}{t}-\omega_j\right)t^{-1}\right]d\omega_1\cdots d\omega_{2m}\Bigg\} \\
&= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[\frac{\cos\left(at\sqrt{1-\sum_{j=1}^{2m}\frac{(x_j-\overline{x}_j)^2}{t^2}}\right)H(t^2-|x-\overline{x}|^2)}{t\sqrt{1-\sum_{j=1}^{2m}\left(\frac{x_j-\overline{x}_j}{t}\right)^2}}\right]\ .
\end{aligned}$$

Hence

$$\begin{aligned}
&\cos\left(\sqrt{-K}\,t\right)(x,\overline{x}) \\
&\quad= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[\frac{\cos\left(a\sqrt{t^2-|x-\overline{x}|^2}\right)}{\sqrt{t^2-|x-\overline{x}|^2}}H(t^2-|x-\overline{x}|^2)\right]\ .
\end{aligned} \tag{4.8}$$

If n is odd, $n = 2m-1$, we get from (2.2) that

$$\cos\left(\sqrt{-K}\,t\right) = \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\Bigg[t^{2m-1}$$

$$\times \int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1} \frac{\cos(a\omega_{2m}t)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}$$

$$\times \prod_{j=1}^{2m-1} \delta(x_j - \overline{x}_j - t\omega_i)d\omega_1\cdots d\omega_{2m}\Bigg]$$

$$= \frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[\int_{-\sqrt{1-\frac{|x-\overline{x}|^2}{t^2}}}^{\sqrt{1-\frac{|x-\overline{x}|^2}{t^2}}} \frac{\cos(at\omega_{2m})}{\sqrt{1-\frac{|x-\overline{x}|^2}{t^2}-\omega_{2m}^2}}d\omega_{2m}\right].$$

Recall that for every $c > 0$,

$$\int_{-c}^{c} \frac{\cos(at\omega)d\omega}{\sqrt{c^2-\omega^2}} = \pi J_0(atc)$$

where J_0 is the Bessel function of the first kind of order 0. Hence

$$\cos\left(\sqrt{-K}\,t\right) = \frac{1}{2(2\pi)^{m-1}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \times \left[J_0\left(a\sqrt{t^2-|x-\overline{x}|^2}\right)H(t^2-|x-\overline{x}|^2)\right]. \tag{4.9}$$

We may continue analytically the formulas (4.8) and (4.9) in a to get (for a purely imaginary) the solution of the initial value problem for the damped wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u + a^2 u$, compare [3].

Example 4.2 *The harmonic oscillator*.

Consider the operator $P = -\frac{d^2}{dx^2} + x^2$. We wish to represent the solution of the initial value problem for the second order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 u}{\partial x^2} + x^2 u \tag{4.10}$$

and for this purpose we express the distribution kernel of $\cos\left(\sqrt{P}\,t\right) = \cos\left(\sqrt{-\frac{d^2}{dx^2}+x^2}\,t\right)$ using the formula (3.4) with $A = \frac{1}{i}\frac{d}{dx}$, $B = x$. Note

that $\cos(B\rho)(x,\overline{x}) = \cos(x\rho)\delta(x-\overline{x})$. Hence (formally)

$$
\begin{aligned}
\cos\left(\sqrt{-\frac{d^2}{dx^2}+x^2\,t}\right) &= \lim_{m\to\infty}\frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[t^{2m-1}\int\cdots\int\right.\\
&\times\int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}\frac{\delta\left(x_1-\overline{x}-\frac{\omega_1 t}{\sqrt{m}}\right)\cos\left(\frac{\omega_2 t}{\sqrt{m}}x_2\right)\delta(x_2-x_1)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}\\
&\times\delta\left(x_3-x_2-\frac{\omega_3 t}{\sqrt{m}}\right)\cos\left(\frac{\omega_4 t}{\sqrt{m}}x_4\right)\delta(x_4-x_3)\\
&\times\delta\left(x_{2m-1}-x_{2m-2}-\frac{\omega_{2m-1}}{\sqrt{m}}\right)\cos\left(\frac{\omega_{2m}t}{\sqrt{m}}x\right)\\
&\left.\times\,\delta(x-x_{2m-1})d\omega_1\cdots d\omega_{2m}dx_1\cdots dx_{2m-1}\right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\cos\left(\sqrt{-\frac{d^2}{dx^2}+x^2+t}\right) &= \lim_{m\to\infty}\frac{1}{(2\pi)^m}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1}\left[t^{2m-1}\right.\\
&\times\int_{\omega_1^2+\cdots+\omega_{2m}^2\le 1}\cos\left[\frac{\omega_2}{\sqrt{m}}\left(\overline{x}+\frac{\omega_1 t}{\sqrt{m}}\right)\right]\cos\left[\frac{\omega_2 t}{\sqrt{m}}\left(\overline{x}+\frac{(\omega_1+\omega_3)t}{\sqrt{m}}\right)\right]\\
&\left.\times\cdots\cos\left(\frac{\omega_{2m}t}{\sqrt{m}}x\right)\delta\left(x-\frac{\omega_1+\omega_3+\cdots+\omega_{2m-1}}{\sqrt{m}}t-\overline{x}\right)d\omega_1\cdots d\omega_{2m}\right].
\end{aligned}
\tag{4.11}
$$

We may interpret (4.11) according to Proposition 3.1, as P is essentially self-adjoint in $L^2(R^1)$, ($\frac{1}{i}\frac{d}{dx}$ and x are self-adjoint in their respective domains). Alternatively, set $H = L^2(R^1; e^{-x^2}dx)$ and let $H_n(x)$ denote the Hermite polynomial $(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$. Then the well-known formulas $H_n'(x) = 2nH_{n-1}(x)$, $xH_n(x) = \frac{H_{n+1}(x)}{2} + nH_{n-1}(x)$ and $\int H_n^2(x)e^{-x^2}dx = \sqrt{\pi}\,2^n n!$ imply that if $h \in \mathrm{Sp}[H_0, H_1, \ldots, H_n]$, then $h \in D\left(A^{\alpha_1}B^{\alpha_2}\cdots A^{\alpha_{2m-1}}B^{\alpha_{2m}}\right)$ for all α and there exist constants C and K such that (3.8) holds. Hence Theorem 3.2 is applicable.

Example 4.3 *Sum of squares of vector fields.*

Let X, Y be vector fields with analytic coefficients defined on an open subset Ω of R^n such that $X^* = -X$, $Y^* = -Y$ (the case of more than two fields may be treated similarly, see Remark 3.4). Let $\varphi(t,\overline{x})$, $\psi(t,\overline{x})$ denote the solutions of the ode's

$$
\begin{aligned}
\frac{d\varphi}{dt}(t,\overline{x}) &= X\varphi(t,\overline{x}),\ \varphi(0,\overline{x}) = \overline{x}\\
\frac{d\psi}{dt}(t,\overline{x}) &= Y\psi(t,\overline{x}),\ \psi(0,\overline{x}) = \overline{x}
\end{aligned}
\tag{4.12}
$$

defined in a neighborhood of $\{0\} \times \Omega$. We have, at least formally, the relation

$$\cos\left(\sqrt{X^2+Y^2}\,t\right)(x,\overline{x}) = \lim_{m\to\infty} \frac{1}{(2\pi)^m} \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \Bigg[t^{2m-1} \int_{\omega_1^2+\cdots+\omega_{2m}^2} \times \frac{\delta\Big(x-\psi\Big(\frac{\omega_{2m}t}{\sqrt{m}}, \varphi\Big(\frac{\omega_{2m-1}t}{\sqrt{m}}, \psi\Big(\frac{\omega_{2m-2}t}{\sqrt{m}}, \dots, \psi\Big(\frac{\omega_2 t}{\sqrt{m}}, \varphi\Big(\frac{\omega_1 t}{\sqrt{m}}, \overline{x}\Big)\cdots\Big)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}} \times d\omega_1\cdots d\omega_{2m} \Bigg] . \tag{4.13}$$

Under further assumptions the integral in the right hand side of (4.13) will be defined for all t. A possible interpretation of (4.13) via Theorem 3.2 is possible along the lines of [5].

Consider the case $X = \frac{\partial}{\partial x_1}$, $Y = x_1\frac{\partial}{\partial x_2}$. Then X^2+Y^2 is a self-adjoint hypoelliptic operator. Moreover, $\varphi(t,\overline{x}) = (\overline{x}_1+t, \overline{x}_2)$, $\psi(t,\overline{x}) = (\overline{x}_1, \overline{x}_2+t)$, and we get from (4.13) that

$$\cos\left(\sqrt{\frac{\partial^2}{\partial x_1^2}+x_1^2\frac{\partial^2}{\partial x_2^2}}\,t\right)(x,\overline{x}) = \lim_{m\to\infty} \frac{1}{(2\pi)^m} \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \Bigg[t^{2m-1} \int_{\omega_1^2+\cdots+\omega_{2m}^2} \times \frac{\delta\Big(x_1-\overline{x}_1-\frac{t}{\sqrt{m}}\Big(\sum_{i=1}^m \omega_{2i-1}\Big), x_2-\overline{x}_2-\frac{t}{\sqrt{m}}\Big(\sum_{i=1}^m \omega_{2i}\Big)\overline{x}_1-\frac{t^2}{m}\sum_{1\le i\le j\le m}\omega_{2i-1}\omega_{2j}\Big)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}} \times d\omega_1\cdots d\omega_{2m} \Bigg] . \tag{4.14}$$

Similarly, set (in R^3) $X = \frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_3}$, $Y = \frac{\partial}{\partial x_2}-2x_1\frac{\partial}{\partial x_3}$. Then $\Delta_H = X^2+Y^2$ is the Laplacian on the Heisenberg group. Here

$$\varphi = (\overline{x}_1+t, \overline{x}_2, \overline{x}_3+2\overline{x}_2 t), \qquad \psi = (\overline{x}_1, \overline{x}_2+t, \overline{x}_3-2\overline{x}_1 t) .$$

The operators X, Y and Δ_H are left invariant. Hence it suffices to compute kernels for $\overline{x} = 0$.

After an elementary calculation, we get from (4.13) that

$$\cos\left(\sqrt{-\Delta_H}\, t\right)(x,0) = \lim_{m\to\infty} \frac{1}{(2\pi)^m} \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{m-1} \Bigg[t^{2m-1} \int_{\omega_1^2+\cdots+\omega_{2m}^2}$$

$$\times \frac{\delta\left(x_1 - \frac{t}{\sqrt{m}}\sum_{i=1}^{m}\omega_{2i-1}, x_2 - \frac{t}{\sqrt{m}}\sum_{i=1}^{m}\omega_{2i}, x_3 - \frac{2t^2}{\sqrt{m}}\left(\sum_{1\le i\le j\le m-1}\omega_{2i}\omega_{2j-1} - \sum_{1\le i\le j\le m}\omega_{2i-1}\omega_{2j}\right)\right)}{\sqrt{1-(\omega_1^2+\cdots+\omega_{2m}^2)}}$$

$$\times\, d\omega_1\cdots d\omega_{2m}\Bigg]. \tag{4.15}$$

The formulas (4.13)–(4.15) may be regarded as "Feynman-Kac" formulas. An explicit evaluation of (4.14) and (4.15) appears to be a challenge.

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